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NUMERICAL-ANALYTICAL INVESTIGATION OF SELF-SIMILAR REGIMES OF PROPAGATION OF NONSTATIONARY CONVECTIVE JETS AND THERMALS IN A HOMOGENEOUS MEDIUM OVER A POINT HEAT SOURCE

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An integral model of a nonstationary vertical convective jet is suggested that involves a universal equation of the propagation of the upper boundary of a convective front depending on the power of the point heat source. A class of self-similar solutions is considered; they correspond to the heat sources whose power changes instantly and also according to the power and exponential laws. Analytical and numerical solutions of the self-similar equations are constructed. Numerical calculations are compared with the well-known experimental data on the profiles of the vertical velocity and temperature on the jet axis.

Theoretical description of nonstationary convective jets has been initiated relatively recently. The first model of such a jet was suggested in [1]. The use in [1] of the vertical boundary-layer approximation and of the integral von Kármán–Pohlhausen method made it possible to construct amplitude equations for a vertical velocity and temperature on the jet axis. The closure of the system of equations was performed with the aid of a heuristic differential equation of transfer for the jet radius. Later, a similar model was used in [2]. Within the framework of the models of [1, 2] that use the modified Taylor's approach, it was possible to construct a class of self-similar solutions corresponding to a point heat source that changes in time according to the power law and also to compare the obtained numerical results with experimental data.

A somewhat different hydrodynamic description of a nonstationary convective jet was developed in [3]. The model was based on the Prandtl approach that prescribed a conical shape of the jet. Moreover, to describe the motion of the plane upper boundary of the jet, which corresponds to the base of the cone, a universal equation of the propagation of a convective front [3, 4] was adopted, which relates the variable height of the jet to the integral time dependence of the power of the point heat source. It is shown in [3, 4] that this model also involves self-similar regimes that correspond to the heat sources changing in time according to power laws. It is essential that the universal equation of the propagation of a convective front allows construction of self-similar solutions that correspond to instantaneous and exponential heat sources that can be considered as enveloping sets of power solutions. Here, the constructed exponential solution represents a self-similar solution of the second kind [5], since it cannot be obtained on the grounds of dimensionality theory.

In the present work, the integral hydrodynamic model of [3, 4] is refined by using the experimentally observed horizontal profiles of temperature and vertical velocity [6]. Within the framework of the integral model, an analytical solution for a convective thermal is constructed which corresponds to instantaneous and point heat sources. A numerical investigation of power and exponential self-similar regimes of the propagation of a nonstationary convective plume is made, which includes comparison with the well-known experi-

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mental data on the profiles of the vertical velocity and temperature on the jet axis. The acceptable agreement of the theory with the observations makes it possible to use the proposed hydrodynamic model in the practice of scientific-engineering calculations of the development of nonstationary convective jets and thermals.

The Problem of a Turbulent Jet over a Point Heat Source. We consider the problem of the propagation of an axisymmetric convective jet in an adiabatic atmosphere over a point heat source. Let t be time and r, φ, z be the cylindrical coordinate system whose z axis is opposite to the free-fall acceleration g.

The convective jet is described with the aid of the Boussinesq convection equation for a perfect gas [7]. Let $\overline{\Theta}_a$ = const be the background static value of the potential temperature of a dry air and Θ be the local potential temperature of the air*). Following [7], we introduce the local dimensionless potential temperature θ:

$$\theta = \frac{\Theta - \overline{\Theta}_a}{\overline{\Theta}_a} . \tag{1}$$

The axisymmetric nonstationary convective jet is described within the framework of the vertical boundary-layer approximation [8]:

$$\frac{\partial}{\partial t} w + \frac{1}{r}\frac{\partial}{\partial r} uwr + \frac{\partial}{\partial z} ww = g\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(\nu_w r \frac{\partial w}{\partial r}\right), \tag{2}$$

$$\frac{\partial}{\partial t} \theta + \frac{1}{r}\frac{\partial}{\partial r} u\theta r + \frac{\partial}{\partial z} w\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu_\theta r \frac{\partial \theta}{\partial r}\right), \tag{3}$$

$$\frac{1}{r}\frac{\partial}{\partial r} ur + \frac{\partial}{\partial z} w = 0 . \tag{4}$$

Equations (2)–(4) can also be used effectively in describing a convective jet in a homogeneous fluid. Here, the quantity θ is understood to be the ratio of the difference between the local density and the density of a nonperturbed fluid to the value of the nonperturbed fluid density.

The system of equations (2)–(4) is considered in the semiinfinite region $V = \{0 \le r \le \infty,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le \infty\}$, where π = 3.14... .

The existence of the kinetic, potential, and internal energy allows an assumption that the functions u^2, w^2, $g\theta z$, and θ belong to the functional space $L_1(V)$ (see [9]). The integrability of the function over the infinite region V leads to the condition of their decay on infinitely distant boundaries.

Taking into account the fact that the medium is not perturbed on the upper boundary of the region, we prescribe the no-flow and flow decay conditions as

$$\lim_{z\to\infty} w(r, z, t) = 0 , \quad \lim_{z\to\infty} 2\pi \int_0^\infty w(r, z, t)\, rdr = 0 , \quad \lim_{z\to\infty} ww(r, z, t) = 0 , \quad \lim_{z\to\infty} 2\pi \int_0^\infty ww(r, z, t)\, rdr = 0 ,$$

$$\lim_{z\to\infty} w\theta(r, z, t) = 0 , \quad \lim_{z\to\infty} 2\pi \int_0^\infty w\theta(r, z, t)\, rdr = 0 . \tag{5}$$

*) The potential temperature of a dry air Θ is defined by the relation $s = c_p \ln(\Theta/\overline{\Theta}_a)$, where s is the entropy of the dry air and c_p is the heat capacity of the dry air at a constant pressure.

As the side boundary-value conditions of system (2)–(4), we adopt the no-flow and flow decay conditions in the form

$$\lim_{r\to\infty} ur = 0\,,\quad \lim_{r\to\infty} \nu_w r\frac{\partial w}{\partial r} = 0\,,\quad \lim_{r\to\infty} \nu_\theta r\frac{\partial \theta}{\partial r} = 0\,. \tag{6}$$

On the lower boundary of the region we prescribe the point nonstationary heat source and zero source of momentum, i.e.,

$$\lim_{z\to 0}\,[ww\,(r, z, t)] = 0\,,\quad \lim_{z\to 0}\,[w\theta\,(r, z, t)] = \frac{1}{2\pi r} S_0\,(t)\,\delta\,(r)\,,\quad S_0\,(t) > 0\,. \tag{7}$$

The state of a nonperturbed atmosphere is taken as the initial condition for $t = t_0$:

$$w\,(r, z, t_0) = 0\,,\quad \theta\,(r, z, t_0) = 0\,. \tag{8}$$

Equations (2)–(8) form a closed system of equations for describing a convective jet.

Integral Model of a Convective Jet over a Point Heat Source. To construct an approximate solution of system (2)–(4), we use the von Kármán–Pohlhausen integral method [8]. Within the framework of this approach, it is assumed that the unknown functions in the region of ascending motion with the characteristic horizontal radius $R(z, t)$ can be approximated by relations with separating variables*):

$$w\,(r, z, t) = \tilde{w}\,(z, t)\,f_w\,(r/R)\,,\quad u\,(r, z, t) = -\frac{\partial \tilde{w}}{\partial z}\,(z, t)\,\frac{1}{r}\int_0^r rf_w\,(r/R)\,dr\,,\quad \theta\,(r, z, t) = \tilde{\theta}\,(z, t)\,f_\theta\,(r/R)\,. \tag{9}$$

Substituting Eq. (9) into Eqs. (2) and (3) and integrating the resulting equations over the area of ascending motions, we can obtain nonstationary amplitude equations of a convective jet at arbitrarily given profiles of f_w and f_θ.

To compare with the already available models [1, 2], we use exponential approximations of horizontal profiles in accordance with the well-known experimental data of [6]:

$$f_w\,(\xi) = \exp\,(-\,\lambda_w \xi^2)\,,\quad f_\theta\,(\xi) = \exp\,(-\,\lambda_\theta \xi^2)\,,\quad \xi = r/R\,. \tag{10}$$

Then, with Eq. (10) taken into account, the corresponding amplitude equations become

$$\frac{\partial}{\partial t}\,\tilde{w}R^2 + \frac{1}{2}\frac{\partial}{\partial z}\,\tilde{w}\tilde{w}R^2 = \alpha_g g\tilde{\theta}R^2\,,\quad \frac{\partial}{\partial t}\,\tilde{\theta}R^2 + \frac{1}{1+\alpha_g}\frac{\partial}{\partial z}\,\tilde{w}\tilde{\theta}R^2 = 0\,,\quad \alpha_g = \lambda_w/\lambda_\theta\,. \tag{11}$$

Substituting Eqs. (9) and (10) into system (7) and integrating the obtained relations over the area of ascending motions, we obtain the boundary conditions for Eq. (11):

$$\lim_{z\to 0}\,[\tilde{w}\tilde{w}R^2\,(z, t)] = 0\,,\quad \lim_{z\to 0}\,[\tilde{w}\tilde{\theta}R^2\,(z, t)] = \frac{1}{\pi}\frac{\lambda_w}{k^2}\,S_0\,(t)\,,\quad k^2 = \frac{\alpha_g}{1+\alpha_g}\,. \tag{12}$$

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*) The existence of the approximations that include spatial separation of variables and similarity over the horizontal coordinates has been reliably established experimentally (see review [10]). In the case of turbulent jets developing at Reynolds numbers Re >> 1 the dependence of the shape of the profiles of f_w and f_θ on the coefficients of turbulent exchange ν_w and ν_θ should be neglected.

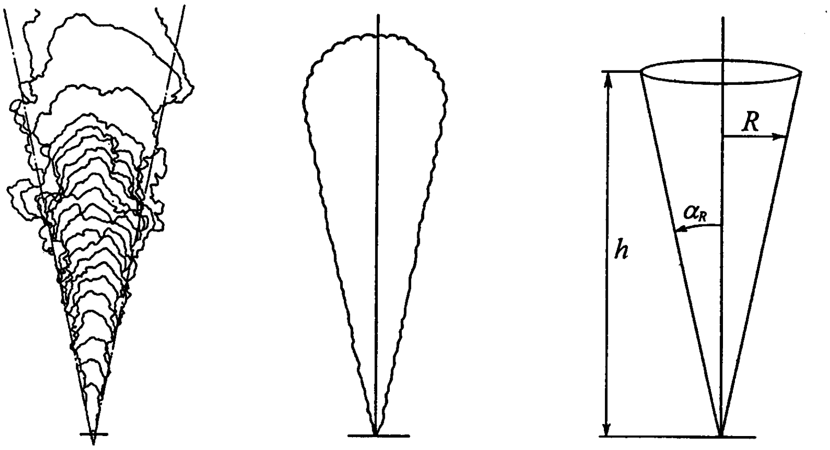

Fig. 1. Contour of a developing convective jet induced by a point heat source, and geometrical stylization of jet propagation.

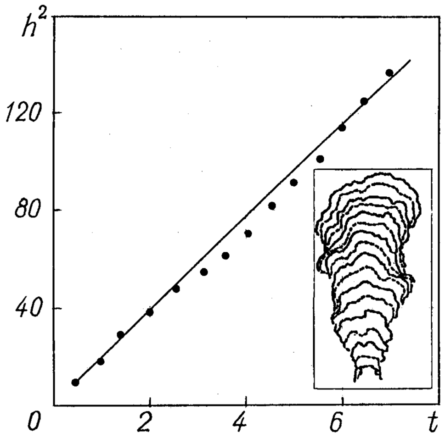

Fig. 2. Propagation of the upper boundary and the contour of the sequential position of the thermal over the point heat source. h^2, m^2; t, sec.

To close the system of equations (11) and (12), it is necessary to include an equation for the radius of the jet R. Following [3, 4], we assume thereafter that the convective thermal is approximated by a conical surface and by a horizontal head part and, moreover, at any time instant the shape of the thermal remains similar (see Fig. 1).

In accordance with the Prandtl hypothesis on linear expansion of a jet adopted in models of [11, 3, 4], for the radius of a plume over a point source we use the relation

$$R(z, t) = \begin{cases} \alpha_R z & \text{for} \quad 0 \le z \le h(t)\,, \\ 0 & \text{for} \quad h(t) \le z \le \infty\,, \end{cases} \tag{13}$$

where α_R is the coefficient of the angular expansion of the jet. The value of this coefficient varies from 0.1 to 0.2 (see, e.g., [10]); $h(t)$ is the time-dependent upper boundary of the conical surface of the jet[*)]

As the equation that describes the propagation of the upper boundary of the convective jet from a heat source in a neutral atmosphere, we adopt the relation suggested in [3, 4]:

*) For details of experimental confirmation of Prandtl's hypothesis, see [12] and also numerical calculations on a multidimensional nonstationary model [13].

$$\lambda_0^2 h^2 \left(\frac{d}{dt} h\right)^2 = g \int_{t_0}^{t} S_0(\tau)\, d\tau , \quad \lambda_0^2 = \frac{\pi}{2} \frac{k^2}{\lambda_w} \alpha_R^2 (1 + \alpha_g)^2 . \tag{14}$$

In subsequent numerical experiments it will be assumed that, according to [6], $\lambda_w = 96\alpha_R^2$, $\lambda_\theta = 0.74\lambda_w$, and, correspondingly, $\alpha_g = \lambda_w/\lambda_\theta = 1.35$. Here, $\lambda_0^2 = 5.15 \cdot 10^{-2}$, in full correspondence with the experimentally observed values $\lambda_0^2 = 2.22 \cdot 10^{-2}$–$4.56 \cdot 10^{-2}$ given in [14] (see Fig. 2).

Relations (11)–(14) form a closed system of equations of the integral model of a vertical convective jet.

Quasistationary Equations as the Asymptotics of the Solution in the Vicinity of a Source. Taking into account the presence of the singularity of the solution in the coordinate origin due to the effect of a point heat source, it is expedient to describe the asymptotic solution of the problem of a convective jet near the source. According to [1, 2], this can be done with the aid of quasistationary equations (11), i.e., equations lacking time derivatives.

Let $\tilde{S}$ be the power of the reduced heat source as prescribed by the relation

$$\tilde{S}(t) = \frac{1}{\pi} \frac{\lambda_w}{k^2} S_0(t) . \tag{15}$$

The velocity, temperature, and radius of the quasistationary jet over the point heat source in an adiabatic atmosphere are

$$\tilde{w}_{a0}(z, t) = \left(\frac{3}{2} \alpha_g g \tilde{S}(t)\right)^{1/3} \alpha_R^{-2/3} z^{-1/3} , \quad \tilde{\theta}_{a0}(z, t) = \left(\frac{3}{2} \alpha_g g \tilde{S}(t)\right)^{-1/3} \tilde{S}(t)\, \alpha_R^{-4/3} z^{-5/3} , \quad R(z, t) = \alpha_R z . \tag{16}$$

Note that relations (16) are also of self-sustained interest, since in the case of a stationary source ($\tilde{S}(t)$ = const) they correspond to an exact solution of the problem of a stationary jet propagating in a neutral atmosphere. The functional relations (16) were first found by Zel'dovich [15] upon application of the similarity theory arguments. Later, relations of the type (15) were obtained as solutions of the integral model of a jet.

Let $h(t)$ be the convective jet height corresponding to the heat source $S_0(t)$ and calculated according to Eq. (14). We introduce the dimensionless variable $z_* = z/h(t)$. Then, the quasistationary solution (16) can be represented as

$$\tilde{w}_{a0}(z, t) = \frac{dh}{dt} w_s^*(z_*, t) , \quad \tilde{\theta}_{a0}(z, t) = \frac{1}{gh} \left(\frac{dh}{dt}\right)^2 \theta_s^*(z_*, t) , \quad R(z, t) = hR_*(z_*, t) , \tag{17}$$

where the dimensionless functions w_s^*, θ_s^*, and R_* and the normalized heat flux power S_0^* have the form

$$w_s^*(z_*, t) = \frac{1}{\alpha_R z_*} \left\{\frac{3}{2} \alpha_g \alpha_R S_0^* z_*^2\right\}^{1/3} , \quad \theta_s^*(z_*, t) = \frac{S_0^*}{\alpha_R z_*} \left\{\frac{3}{2} \alpha_g \alpha_R S_0^* z_*^2\right\}^{-1/3} ,$$

$$R_*(z_*, t) = \alpha_R z_* , \quad S_0^*(t) = \frac{1}{\pi} \frac{\lambda_w}{k^2} g S_0(t) \Big/ \left(h \left(\frac{dh}{dt}\right)^3\right) . \tag{18}$$

It is obvious that the functions (18) satisfy the relations

$$\frac{1}{2}\frac{\partial}{\partial z_*} w_s^* w_s^* R_*^2 = \alpha_g \theta_s^* R_*^2 , \quad \frac{1}{1+\alpha_g}\frac{\partial}{\partial z_*} w_s^* \theta_s^* R_*^2 = 0 . \tag{19}$$

Next, for comparison with experimental data the general solution of the nonstationary problem will be normalized to the quasistationary solution (16)–(18).

Self-Similar Regimes of Propagation of a Convective Front. For a number of specially prescribed amplitudes of a heat source the corresponding regimes of propagation of a convective front can be obtained on the basis of the dimensionality theory without resorting to the proposed universal relation (14). We will show that Eq. (14) not only includes all of the earlier known relations as specific cases, but also allows one to construct new self-similar regimes.

We consider the convection caused by an instantaneous heat source. Then, Eq. (14) yields

$$t_0 = 0 , \quad S_0(t) = Q_0\, \delta(t) , \quad Q_0 = \text{const} , \quad h(t) = \left(\frac{2}{\lambda_0}\right)^{1/2} (gQ_0)^{1/4}\, t^{1/2} . \tag{20}$$

The relation corresponding to Eq. (20) was obtained by the similarity and dimensional analysis method and was checked experimentally in [14] (see Fig. 2).

Let us consider the convection caused by the power heat source. Then, Eq. (14) for $t_0 = 0$, $S_0(t) = Q_q q t^{q-1}$, $q > 0$, Q_q = const yields

$$h(t) = \left\{\left(\frac{2}{\lambda_0}\right)^2 \frac{gQ_q}{(q/2+1)^2}\, t^{(q+2)}\right\}^{1/4} . \tag{21}$$

The relation corresponding to Eq. (21) was obtained by the similarity and dimensional analysis method and was checked experimentally in [1, 2].

Now, we consider the convection caused by the exponential heat source. Then, Eq. (14) for $t_0 = -\infty$, $S_0(t) = Q_\infty q \exp(qt)$, and Q_∞ = const yields

$$h(t) = \left\{\left(\frac{2}{\lambda_0}\right)^2 \frac{gQ_\infty}{(q/2)^2} \exp(qt)\right\}^{1/4} . \tag{22}$$

Formula (22) is a self-similar relation of the second kind [5], since it cannot be obtained only on the basis of dimensionality-theory arguments. It should be interpreted as the "envelope" of a set of power solutions (21) for $t_0 \to -\infty$ and $q \to +\infty$.

Development of Self-Similar Jets over a Point Heat Source. The integral model of a nonstationary jet and the corresponding self-similar solutions for point heat sources have been constructed for the first time in [1, 2]. We will show that for point heat sources the self-similar regimes (20)–(22) generate the corresponding classes of self-similar motions also for the proposed integral jet model.

Let $z_* = z/h(t)$ be a dimensionless parameter. The self-similar solution of system (11)–(14) for $0 < z < h(t)$ can be sought in the form

$$\tilde{w}(z, t) = \frac{dh}{dt} w^*(z_*) , \quad \tilde{\theta}(z, t) = \frac{1}{gh}\left(\frac{dh}{dt}\right)^2 \theta^*(z_*) , \quad R(z, t) = hR_* = h\alpha_R z_* . \tag{23}$$

For power sources (21), the substitution of Eq. (23) into system (11) for $0 < z_* < 1$ leads to an system of ordinary differential equations:

$$\left[3+\frac{q-2}{q+2}\right]w^*R_*^2-\frac{d}{dz_*}(w^*R_*^2z_*)+\frac{1}{2}\frac{d}{dz_*}w^*w^*R_*^2=\alpha_g\theta^*R_*^2\,,$$

$$2\left[1+\frac{q-2}{q+2}\right]\theta^*R_*^2-\frac{d}{dz_*}(\theta^*R_*^2z_*)+\frac{1}{1+\alpha_g}\frac{d}{dz_*}w^*\theta^*R_*^2=0\,,\quad R_*=\alpha_Rz_*\,. \tag{24}$$

According to Eq. (12), the boundary-value conditions of system (24) have the form

$$\lim_{z_*\to 0}\left\{w^*w^*R_*^2\right\}=0\,,\quad \lim_{z_*\to 0}\left\{w^*\theta^*R_*^2\right\}=\frac{1}{\pi}\frac{\lambda_w}{k^2}\frac{4q}{q+2}\lambda_0^2\,. \tag{25}$$

Similar relations can be derived for an instantaneous (20) and an exponential (22) source. Here, the coefficients in corresponding equations (24) and (25) result from the limiting transition for $q\to 0$ and $q\to\infty$.

We note that in the region of large enough exponents of the sources $20<q<\infty$ the coefficients (25) can be considered constant, independent of q. Thus, all the self-similar jets with large enough exponents of sources have practically identical velocity and temperature profiles that correspond to the exponential source.

It should be emphasized that systems of equations of the type (24) and (25) will hold in the case of arbitrarily specified profiles f_w and f_θ. Consequently, the self-similarity constructed is the common property of the one-dimensional nonstationary model.

Analytical Description of the Self-Similar Regime of the Development of a Thermal. For the case of an instantaneous point heat source (20) the nonzero analytical solution of (24) and (25) at $q = 0$ is

$$w^*(z_*)=(1+\alpha_g)\,z_*\,,\quad \theta^*(z_*)=2\,(1+\alpha_g)\,z_*\,,\quad R_*(z_*)=\alpha_Rz_*\,. \tag{26}$$

We show that analytical solution (26) implements the integral heat balance that corresponds to instantaneous heat generation. Indeed, let $\xi = r/R$ and $z_* = z/h(t)$; then the dimensionless potential temperature of the thermal θ can be represented, with account for (9), (10), and (20), in the form

$$\theta\,(r,z,t)=\tilde{\theta}\,(z,t)\,f_\theta\,(\xi)=\frac{1}{gh}\left(\frac{dh}{dt}\right)^2\theta^*(z_*)\,f_\theta\,(\xi)=\left(\frac{Q_0}{\lambda_0^2h^3}\right)\theta^*(z_*)\exp\,(-\lambda_\theta\xi^2)\,. \tag{27}$$

Thus, the total temperature of the propagating layer, calculated with account for Eqs. (26), (27), and (13), satisfies the integral relation of instantaneous heat generation:

$$2\pi\int_0^h\int_0^\infty\theta r drdz=\frac{2\pi\alpha_R^2}{2\lambda_0}\int_0^h\tilde{\theta}\,(z,t)\,z^2dz=\frac{\pi\alpha_R^2\alpha_g}{\lambda_w}\left(\frac{Q_0}{\lambda_0^2h^3}\right)h^3\int_0^1\theta^*(z_*)\,z_*^2dz_*=$$

$$=2\,(1+\alpha_g)\frac{\pi\alpha_R^2\alpha_g}{\lambda_0^2\lambda_w}Q_0\int_0^1 z_*^3dz_*=\frac{(1+\alpha_g)}{2}\frac{\pi\alpha_R^2\alpha_g}{\lambda_0^2\lambda_w}Q_0=Q_0\,. \tag{28}$$

Analytical solution (26) shows that the velocity and temperature on the axis of the convective thermal increase linearly with height. To be sure, in an actually developing convective thermal the linear increase in the velocity and potential temperature is maintained just to a certain level $0\le z_*<0.75$, above which the conical shape of the thermal ceases to exist.

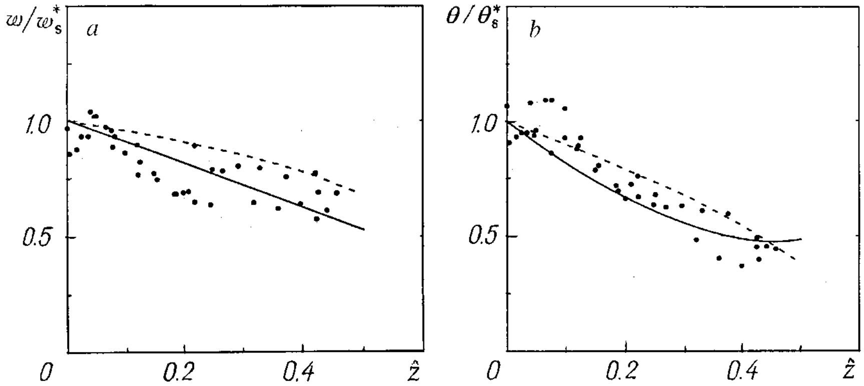

Fig. 3. Calculation of the dependence of the normalized vertical velocity w/w_s^* (a) and potential temperature θ/θ_s^* (b) on $\hat{z}$ (solid lines) and also experimental data corresponding to different runs of measurements [2] (dots) and numerical calculations by the model of [2] (dashed lines).

Relations (26) imply the existence of a dimensionless parameter which is constant along the axis of the thermal that rises in a neutral atmosphere:

$$\frac{\tilde{w}^2}{g\tilde{\theta}R} = \frac{(w^*)^2}{\theta^* R_*} = \frac{1+\alpha_g}{2\alpha_R} . \quad (29)$$

This expression should be considered as the theoretical foundation of Scorer's empirical relation [14] obtained for air bubbles rising in a homogeneous liquid, with the right-hand side of Eq. (29) being independent of either the height or the power of the instantaneous source.

Numerical Description of Self-Similar Regimes of Jet Development. Taking into account the asymptotic behavior of a convective jet, when $z_* << 1$, in accordance with [1, 2] the subsequent representations of the results will use the functions φ_w and φ_θ, where

$$\frac{w}{w_{a0}} = \frac{w^*}{w_s^*} = \varphi_w (z_*) , \quad \lim_{z_* \to 0} \varphi_w (z_*) = 1 ; \quad \frac{\theta}{\theta_{a0}} = \frac{\theta^*}{\theta_s^*} = \varphi_\theta (z_*) , \quad \lim_{z_* \to 0} \varphi_\theta (z_*) = 1 . \quad (30)$$

Following [1, 2], we introduce the dimensionless parameter $\hat{z}$, where

$$\hat{z} = \frac{1}{S_0 (t)} \frac{dS_0}{dt} \frac{z}{\tilde{w}_{a0} (z, t)} . \quad (31)$$

Using relations (15) and (16), we may show that for exponential heat sources

$$\hat{z} = C z_*^{4/3} , \quad C = \frac{4}{(1+\alpha_g)} \left(\frac{q-1}{q+2} \right) \left(\frac{q+2}{3q} \right)^{1/3} . \quad (32)$$

We consider, as an example, numerical solution of self-similar equations for the case of $q = 4$ for the involvement coefficient $\alpha_R = 0.1$. The results of calculations of the normalized velocity and potential temperature and their comparison with the experimental data of [2] are presented in Fig. 3.

Conclusions. The analytical results of the present work show that the proposed integral model of a convective jet that includes a universal equation for the propagation of the upper boundary of a convective front contains a class of self-similar solutions corresponding to instantaneous, power, and exponential heat sources. The results of calculations of the vertical velocity and potential temperature profiles on the jet axis by the proposed model point to the acceptable correlation with both the well-known experimental data and theoretical models [1, 2].

Appendix. Equation for Convective Front Propagation above a Point Heat Source. According to [16], we determine the convective front as a mobile horizontal plane $z = h(t)$, on the σ surface of which

$$\frac{dh}{dt}\int_{\sigma} \theta\,(r, h, t)\, d\sigma - \int_{\sigma} w\theta\,(r, h, t)\, d\sigma = 0\,, \quad d\sigma = 2\pi r dr\,. \tag{A.1}$$

We show in what follows that a layer of atmospheric air located above the convective front $z = h(t)$ is not heated at all by the point heat source.

Direct experimental determination of the convective front height from (A.1) is rather difficult. Only the change in the upper boundary of the colored convective plume that corresponds to zero dimensionless pulsation of the potential temperature θ is accessible for direct measurements in practice.

We approximate a convective plume over a point heat source by a cone of equivalent volume (see Fig. 1). Then, on the upper boundary of the conical surface $\theta = 0$ and, consequently, (A.1) is satisfied identically.

Within the framework of approximation (9) and (13), on the surface of the convective front $z = h(t)$

$$w\,(r, h, t) = \tilde{w}\,(h, t)\, f_w\,(r/R)\,, \quad \theta\,(r, h, t) = \tilde{\theta}\,(h, t)\, f_{\theta}\,(r/R)\,, \quad R\,(h, t) = \alpha_R h\,(t)\,. \tag{A.2}$$

Substituting (A.2) into (A.1), we obtain

$$(1 + \alpha_g)\,\frac{dh}{dt} = \tilde{w}\,(h, t)\,, \tag{A.3}$$

i.e., the velocity of the front is determined only by the velocity of the particles located on the jet axis.

Let us pass to the construction of the basic integral relations. The integration of Eq. (4) with account for the boundary-value conditions (6) and (5) yield

$$\frac{\partial}{\partial z}\int_{\sigma} w d\sigma = 0\,, \quad \int_{0}^{h(t)}\int_{\sigma} w d\sigma dz = 0\,. \tag{A.4}$$

Integrating Eqs. (2) and (3) with account for boundary conditions (6) and (7), we obtain

$$\int_{\sigma} w^2\,(r, h, t)\, d\sigma = g \int_{0}^{h(t)}\int_{\sigma} \theta d\sigma dz\,, \tag{A.5}$$

$$\frac{\partial}{\partial t}\int_{0}^{h(t)}\int_{\sigma} \theta d\sigma dz - \left\{\frac{dh}{dt}\int_{\sigma} \theta\,(r, h, t)\, d\sigma - \int_{\sigma} w\theta\,(r, h, t)\, d\sigma\right\} = S_0\,(t)\,. \tag{A.6}$$

Integration of relation (A.6) with account for initial conditions (8) and Eq. (A.1) and substitution of the obtained result into (A.5) yields

$$2\pi \int_0^\infty w^2 (r, h, t)\, r dr = g \int_{t_0}^{t} S_0 (\tau)\, d\tau \, . \tag{A.7}$$

The use of (A.2) and (A.3) in (A.7) leads to the equation for the propagation of the convective front over the point source of buoyancy:

$$\lambda_0^2 h^2 \left(\frac{d}{dt} h\right)^2 = g \int_{t_0}^{t} S_0 (\tau)\, d\tau \, , \quad \lambda_0^2 = \frac{\pi}{2\lambda_w} \alpha_R^2 (1 + \alpha_g)^2 \, . \tag{A.8}$$

The value of the constant $\lambda_0^2 = 9.04 \cdot 10^{-2}$ calculated from Eq. (10) somewhat exceeds the experimental values $\lambda_0^2 = 2.22 \cdot 10^{-2}$–$4.56 \cdot 10^{-2}$ obtained in ejection of colored liquid into fresh water and given in [11]. Taking into account the approximate character of the boundary-layer equations and of the von Kármán–Pohlhausen integral method, the value λ_0^2 derived should not be considered as an exact value that characterizes the actual propagation of a convective front. Therefore, in the calculations given above it is advisable to use the somewhat smaller value $\lambda_0^2 = \frac{\pi}{2} \frac{k^2}{\lambda_w} \alpha_R^2 (1 + \alpha_g)^2 = 5.15 \cdot 10^{-2}$, which agrees much better with the experimental data of [14].

Equation (A.8) admits a rather clear physical interpretation. It is obvious that the effective area of the convective front is proportional to h^2; therefore, the value $h^2 (dh/dt)^2$ corresponds to the total kinetic energy of the convective front (see the left-hand side of (A.8)). Thus, according to (A.8), the total kinetic energy of the convective front at any instant of time is proportional to the value of the work of buoyancy forces which comes to the medium from the source on the substrate surface. The modification of Eq. (A.8) to the case of a linear and plane heat source is given in [4].

NOTATION

u and w, components of velocity along the r and z axes, respectively, m/sec; $\nu_{\rm w}$ and ν_θ, coefficients of turbulent exchange for a vertical velocity and dimensionless potential temperature, m^2/sec; S_0, power of point source of heat, m^3/sec; $\delta(r)$, Dirac delta-function; $\tilde{w}$ and $\tilde{\theta}$, vertical velocity and dimensionless potential temperature on the axis of a nonstationary jet, m/sec; $\tilde{w}_{a0}$ and θ_{a0}, amplitudes of the vertical velocity and dimensionless potential temperature on the axis of a quasistationary jet, m/sec; f_w and f_θ, dimensionless horizontal profiles of vertical velocity and potential temperature; λ_w and λ_θ, numerical parameters characterizing dimensionless horizontal profiles of the vertical velocity and potential temperature; w^*, θ^*, R_*, and S_0^*, normalized dimensionless functions of the vertical velocity, potential temperature, radius, and power of a heat source; w_s^* and θ_s^*, normalized dimensionless functions of the vertical velocity and potential temperature of a quasistationary jet.

REFERENCES

1. M. A. Delichatsios, *J. Fluid Mech.*, **93**, Pt. 2, 241–250 (1979).
2. Yu. Hong-Zeng, *J. Heat Transfer*, **112**, 186–191 (1990).
3. A. N. Vul'fson, *Neftepromysl. Delo*, No. 8, 45–48 (1999).
4. A. N. Vul'fson, *Izv. Ross. Akad. Nauk, Fiz. Atmos. Okeana*, **34**, No. 4, 557–564 (1998).
5. G. I. Barenblatt, *Similarity, Self-Similarity, Intermediate Asymptotics* [in Russian], Leningrad (1982).

6. H. Rouse, C.-S. Jih, and H. W. Humphreys, *Tellus*, **4**, No. 3, 201–210 (1952).
7. Y. Ogura and N. A. Phillips, *J. Atmos. Sci.*, **19**, No. 2, 173–179 (1962).
8. H. Schlichting, *Boundary Layer Theory*, London (1955).
9. W. Rudin, *Principles of Mathematical Analysis*, New York–London (1964).
10. G. N. Abramovich, *Theory of Turbulent Jets* [in Russian], Moscow (1984).
11. F. H. Schmidt, *Tellus*, **9**, No. 3, 378–383 (1957).
12. V. Andreev and S. Panchev, *Dynamics of Atmospherical Thermals* [in Russian], Leningrad (1975).
13. O. M. Belotserkovskii, V. A. Andrushchenko, and Yu. D. Shevelev, *Dynamics of Spatial Vortex Flows in an Inhomogeneous Atmosphere* [in Russian], Moscow (2000).
14. R. S. Scorer, *Environmental Aerodynamics*, New York–London–Sydney–Toronto (1978).
15. Ya. B. Zel'dovich, *Zh. Éksp. Teor. Fiz.*, **7**, Issue 12, 1463–1465 (1937).
16. T. Yamada and S. Berman, *J. Appl. Meteor.*, **18**, No. 6, 781–786 (1979).